arXiv:math/0404416v2 [math.DS] 16 May 2004

There are no structurally stable diffeomorphisms of odd-dimensional manifolds with codimension one non-orientable expanding attractors

V. Medvedev E. Zhuzhoma

Dedicated to Carlos Gutierrez on his 60th birthday

Abstract

We prove that a structurally stable diffeomorphism of closed $(2m+1)$-manifold, $m \geq 1$, has no codimension one non-orientable expanding attractors.

1 Introduction

Structurally stable diffeomorphisms exist on any closed manifold (say a diffeomorphism f structurally stable if all diffeomorphisms C^1-close to f are conjugate to f). It is natural to study the question of existence of such diffeomorphisms with some additional conditions. The condition we consider here is the presence of a codimension one non-orientable expanding attractor. Due to well known example of Plykin [7], the answer is YES for 2-manifolds. Medvedev and Zhuzhoma [6] proved that for 3-manifolds the answer is NO. In the paper, we generalize the result of [6] proving that there are no structurally stable diffeomorphisms with a codimension one non-orientable expanding attractor on closed odd-dimensional manifolds. The proof is shorter than [6] and includes $d = 3$. As to orientable attractors, the answer is YES for any $d \geq 2$. Namely, starting with a codimension one Anosov diffeomorphism of the d-torus T^d, $d \geq 2$, a structurally stable diffeomorphism of T^d with an

orientable codimension one expanding attractor can be obtained by Smale's surgery [11], so-called DA-diffeomorphism (see also [4], [8], [10]).

Before the formulation of exact result, we give necessary definitions and notions. Let $f : M \to M$ be a diffeomorphism of a closed d-manifold M, $d = \dim M \geq 2$, endowed with some Riemann metric ρ (all definitions in this section can be found in [4] and [10], unless otherwise indicated). A point $x \in M$ is *non-wandering* if for any neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for infinitely many integers n. Then the non-wandering set $NW(f)$, defined as the set of all non-wandering points, is an f-invariant and closed. A closed invariant set $\Lambda \subset M$ is *hyperbolic* if there is a continuous f-invariant splitting of the tangent bundle $T_\Lambda M$ into stable and unstable bundles $E^s_\Lambda \oplus E^u_\Lambda$ with

$$\|df^n(v)\| \leq C\lambda^n \|v\|, \quad \|df^{-n}(w)\| \leq C\lambda^n \|w\|, \quad \forall v \in E^s_\Lambda, \forall w \in E^u_\Lambda, \forall n \in \mathbb{N},$$

for some fixed $C > 0$ and $\lambda < 1$. For each $x \in \Lambda$, the sets $W^s(x) = \{y \in M : \lim_{j\to\infty} \rho(f^j(x), f^j(y)) \to 0$, $W^u(x) = \{y \in M : \lim_{j\to\infty} \rho(f^{-j}(x), f^{-j}(y)) \to 0$ are smooth, injective immersions of E^s_x and E^u_x that are tangent to E^s_x, E^u_x respectively. $W^s(x)$, $W^u(x)$ are called *stable* and *unstable manifolds* at x.

For a diffeomorphism $f : M \to M$, Smale [11] introduced the Axiom A: $NW(f)$ is hyperbolic and the periodic points are dense in $NW(f)$. A diffeomorphism satisfying the Axiom A is called A-diffeomorphism. According to Spectral Decomposition Theorem, $NW(f)$ of an A-diffeomorphism f is decomposed into finitely many disjoint so-called basic sets $B_1, \ldots, B_k$ such that each B_i is closed, f-invariant and contains a dense orbit.

A basic set Ω is called an *expanding attractor* if there is a closed neighborhood U of Ω such that $f(U) \subset int\ U$, $\cap_{j\geq 0} f^j(U) = \Omega$, and the topological dimension $\dim \Omega$ of Ω is equal to the dimension $\dim(E^u_\Omega)$ of the unstable splitting E^u_Ω (the name is suggested in [12], [13]). Ω is codimension one if $\dim\ \Omega = \dim\ M - 1$. It is well known that a codimension one expanding attractor consists of the $(d-1)$-dimensional unstable manifolds $W^u(x)$, $x \in \Omega$, and is locally homeomorphic to the product of $(d-1)$-dimensional Euclidean space and a Cantor set. $W^s(x)$ is homeomorphic to $\mathbb{R}$ and can be endowed with some orientation. $W^u(x)$ is homeomorphic to $\mathbb{R}^{d-1}$ and can be endowed with some normal orientation (even if M is non-orientable). Due to hyperbolic structure, any $W^s(x)$ intersects $W^u(x)$ transversally, $x \in \Omega$. Following [1], say that Ω is *orientable* if for every $x \in \Omega$ the index of the intersection $W^s(x) \cap W^u(x)$ does not depend on a point of this intersection (it is either $+1$ or -1). The main result is the following theorem.

Theorem 1 *Let $f : M \to M$ be a structurally stable diffeomorphism of a closed $(2m+1)$-manifold M, $m \geq 1$. Then the spectral decomposition of f does not contain codimension one non-orientable expanding attractors.*

Our proof does not work for even-dimensional manifolds for which the existence of codimension one non-orientable expanding attractors stay open question (except $d = 2$).

Acknowledgment. The research was partially supported by RFFI grant 02-01-00098. We thank Roman Plykin and Santiago Lopez de Medrano for useful discussions. This work was done while the second author was visiting Rennes 1 University (IRMAR) in March-Mai 2004. He thanks the support CNRS which made this visit possible. He would like to thank Anton Zorich and Vadim Kaimanovich for their hospitality.

2 Proof of the main theorem

Later on, Ω is a codimension one non-orientable expanding attractor of diffeomorphism $f : M \to M$. A point $p \in \Omega$ is called *boundary* if at least one component of $W^s(p) - p$ does not intersect Ω. Boundary points exist and satisfy to the following conditions [1], [7]:

- There are finitely many boundary points and each is periodic.
- Given a boundary point $p \in \Omega$, there is a unique component of $W^s(p) - p$ denoted by $W^s_\emptyset(p)$ which does not intersect Ω.
- Given a point $x \in W^u(p) - p$, there is a unique arc $(x, y)^s \subset W^s(x)$ denoted by $(x, y)^s_\emptyset$ such that $(x, y)^s \cap \Omega = \emptyset$ and $y \in \Omega$.

An unstable manifold $W^u(p)$ containing a boundary point is called a *boundary unstable manifold*. Due to [1] and [8], the accessible boundary of $M - \Omega$ from $M - \Omega$ is a finite union of boundary unstable manifolds that splits into so-called bunches defined as follows. The family $W^u(p_1), \ldots, W^u(p_k)$ is said to be a *k-bunch* if there are points $x_i \in W^u(p_i)$ and arcs $(x_i, y_i)^s_\emptyset$, $y_i \in W^u(p_{i+1})$, $1 \leq i \leq k$, where $p_{k+1} = p_1$, $y_k \in W^u(p_1)$, and there are no $(k+1)$-bunches containing the given one.

Lemma 1 *Let $f : M \to M$ be an A-diffeomorphism of a closed $(2m+1)$-manifold M, $m \geq 1$. If the spectral decomposition of f contains a codimension one non-orientable expanding attractor, then M is non-orientable.*

Proof. The non-orientability of Ω implies that Ω has at least one 1-bunch, say $W^u(p)$ [8]. Therefore, given any point $x \in W^u(p) - p$, there is a unique point $y \in W^u(p) - p$ such that $(x,y)^s = (x,y)^s_\emptyset$, and vise versa. Let the map $\phi : W^u(p) - p \to W^u(p) - p$ be given by $\phi(x) = y$ whenever $(x,y)^s = (x,y)^s_\emptyset$. Then ϕ is an involution, $\phi^2 = id$.

Let r be the period of p. Since the stable (as well as unstable) manifolds are f-invariant, $f^r \circ \phi|_{W^u(p)} = \phi \circ f^r|_{W^u(p)}$. Since the restriction $f^r|_{W^u(p)}$ is an expantion map with the unique hyperbolic fixed point p, ϕ can be extended homeomorphically to $W^u(p)$ putting $\phi(p) = p$. By theorem 2.7 and lemma 2.1 [8], ϕ is conjugate to the antipodal involution, i.e. there exist a homeomorphism $h : W^u(p) \to \mathbb{R}^{d-1}$ (in the intrinsic topology of $W^u(p)$) and the involution $\theta : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ of the type $\vec{v} \to -\vec{v}$ such that $\theta \circ h = h \circ \phi$. This implies that there is the $(d-1)$-dimensional ball $B^{d-1} \subset W^u(p)$ such that $p \in B^{d-1}$, the boundary $\partial B^{d-1} \stackrel{\text{def}}{=} S^{d-2}$ is tamely embedded in $W^u(p)$, and S^{d-2} is ϕ-invariant. Moreover, there is the annulus $S^{d-2} \times [0,1] \subset W^u(p)$ foliated by $S^{d-2}_t = S^{d-2} \times \{t\}$, $t \in [0,1]$, $S^{d-2} = S^{d-2}_0$, such that every S^{d-2}_t is ϕ-invariant and bounds the $(d-1)$-dimensional ball $B^{d-1}_t \subset W^u(p)$ containing p. Since $\phi^2 = id$, the set

$$B^{d-1}_t \bigcup_{x \in S^{d-2}_t} [x, \phi(x)] \stackrel{\text{def}}{=} P_t$$

is homeomorphic to the projective space $\mathbb{R}P^{d-1}$ for every $t \in [0,1]$. Since $d - 1 = 2m$ is even, P_t is non-orientable. For any $x \in S^{d-2}_{t_1}$ and $y \in S^{d-2}_{t_2}$ with $t_1 \neq t_2$, $[x, \phi(x)]^s_\infty \cap [y, \phi(y)]^s_\infty = \emptyset$. Hence the set

$$\bigcup_{x \in S^{d-2} \times [0,1]} [x, \phi(x)] \subset M$$

is homeomorphic to $\mathbb{R}P^{d-1} \times [0,1]$. Since $\mathbb{R}P^{d-1} \times [0,1]$ is a non-orientable d-manifold, M is non-orientable. $\square$

Proof of theorem 1. Assume the converse. Then the spectral decomposition of f contains a codimension one non-orientable expanding attractor, say Ω. According to lemma 1, M is non-orientable. Let $\overline{M}$ be an orientable manifold such that $\pi : \overline{M} \to M$ is a (nonbranched) double covering for M. Then there exists a diffeomorphism $\overline{f} : \overline{M} \to \overline{M}$ that cover f, i.e., $f \circ \pi = \pi \circ \overline{f}$. It is easy to see that $\overline{f}$ is an A-diffeomorphism with a codimension one expanding attractor $\overline{\Omega} \subset \pi^{-1}(\Omega)$. It follows from lemma 1 and orientability of $\overline{M}$ that $\overline{\Omega}$ is orientable.

Because of f is a structurally stable diffeomorphism, f satisfies to the strong transversality condition [5] which is a local condition. Since π is a local diffeomorphism, $\overline{f}$ satisfies to the strong transversality condition as well. Hence, $\overline{f}$ is structurally stable [9].

Take a periodic point $p \in \Omega$ on the boundary unstable manifold $W^u(p)$ that is a 1-bunch. Then the preimage $\pi^{-1}(W^u(p))$ is a 2-bunch of $\overline{\Omega}$ consisting of unstable manifolds $W^u(p_1)$, $W^u(p_2)$, where $\{p_1, p_2\} = \pi^{-1}(p)$ are boundary periodic points of $\overline{f}$. It was proved in [2], [3] that $W^s_\emptyset(p_1)$ and $W^s_\emptyset(p_2)$ belong to the unstable manifolds $W^u(\alpha_1)$ and $W^u(\alpha_2)$ respectively of the repelling periodic points α_1, α' (possibly, $\alpha_1 = \alpha'$). Moreover, there are repelling periodic points $\alpha_1, \ldots, \alpha_{k+1} = \alpha'$ and saddle periodic points $P_1 = p_1$, P_2, $\ldots, P_{k+1}$, $P_{k+2} = p_2$, $k \geq 0$, of index $d-1$ such that the following conditions hold:

1. The set
$$l = P_1 \cup W^s_\emptyset(P_1) \cup \alpha_1 \cup W^s(P_2) \cup \ldots \cup \alpha_{k+1} \cup W^s_\emptyset(P_{k+2}) \cup P_{k+2}$$
is homeomorphic to an arc with no self-intersections whose endpoints are P_1 and P_{k+2}.

2. $l - (P_1 \cup P_{k+2}) \subset \overline{M} - \overline{\Omega}$.

3. The repelling periodic points α_i alternate with saddle periodic points P_i on l.

It follows from $f \circ \pi = \pi \circ \overline{f}$ that π maps the stable and unstable manifolds of $\overline{f}$ into the stable and unstable manifolds respectively of f. Since $\pi(P_1) = \pi(P_{k+2}) = p$,
$$\pi(W^s_\emptyset(P_1)) = \pi(W^s_\emptyset(P_2)), \quad \pi(\alpha_1) = \pi(\alpha_{k+1}).$$
Hence (if $k \geq 1$),
$$\pi(W^s(P_2)) = \pi(W^s(P_{k+1})), \quad \pi(P_2) = \pi(P_{k+1}), \quad \pi(\alpha_2) = \pi(\alpha_k), \quad \ldots.$$
Due to item (3) above, the number of all periodic points on l equals $2k+3$ that is odd. As a consequence, there is either a periodic point α_i with $\pi(W^s(P_i)) = \pi(W^s(P_{i+1}))$ or a periodic point P_i with $\pi(W^s_1(P_i)) = \pi(W^s_2(P_i))$, where $\pi(W^s_1(P_i))$, $\pi(W^s_2(P_i))$ are different components of $W^s(P_i) - P_i$. In both cases, there is a point (α_i or P_i) at which π is not a local homeomorphism. This contradiction concludes the proof. $\square$

References

[1] **Grines V.** On topological conjugacy of diffeomorphisms of a two-dimensional manifold onto one-dimensional basic sets I, *Trans. Moscow Math. Soc.*, **32**(1975), 31-56. MR **54:** 6203

[2] **Grines V., Zhuzhoma E.** On structurally stable diffeomorphisms with expanding attractors or contracting repellers of codimension one. *Dokl. RAN*, **374**(2000), 274-276 (Russian). Transl.: *Dokl. Akad. Nauk*, **374**(2000), no 6, 735-737. MR **2002b:** 37038.

[3] **Grines V., Zhuzhoma E.** Structurally stable diffeomorphisms with codimension one basic sets. Preprint, University de Bourgogne (Dijon), no 223, **2000**.

[4] **Katok A., Hassenblatt B.** *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Math. and its Appl., Cambridge University Press, **1994**. MR **96:** 58055

[5] **Mañé R.** A proof of C^1 stability conjecture. *Publ. IHES*, **66**(1988), 161-210. MR **89e:** 58090

[6] **Medvedev V., Zhuzhoma E.** Structurally stable diffeomorphisms have no codimension one Plykin attractors on 3-manifolds. Proc. of *Foliations: Geometry and Dynamics* (held in Warsaw, May 29 - June 9, 2000), World Scientific, Singapore, **2002**, 355-370. MR **2002k:** 37036

[7] **Plykin R.V.** Sources and sinks of A-diffeomorphisms of surfaces. *Mat. Sb.*, **94**(1974), 243-264 (Russian). Transl.: *Mathem. in USSR, Sbornik*, **23**(1974), 233-253, MR 0356137 (50 $\sharp$ 8608).

[8] **Plykin R.V.** On the geometry of hyperbolic attractors of smooth cascades. *Usp. Math. Nauk*, **39**(1984), 75-113 (Russian). Transl.: *Russian Math. Surveys*, **39**(1984), no 6, 85-131. MR **86f:** 58106.

[9] **Robinson C.** Structural stability of C^1 diffeomorphisms. *Journ. Diff. Equat.*, **22**(1976), 28-73. MR **57:** 14051

[10] **Robinson C.** *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos*, second edition. CRC Press, **1999**. MR **2001k:** 37003

[11] **Smale S.** Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 1967, 73, 1, 741-817. MR **37:** 3598

[12] **Williams R.F.** One-dimensional non-wandering sets. *Topology*, **6**(1967), 473-487. MR **36:** 897

[13] **Williams R.F.** Expanding attractors. *Publ. Math., IHES*, **43**(1974), 169-203. MR **50:** 1289

Dept. of Diff. Equat., Inst. of Appl. Math. and Cyber., Nizhny Novgorod State University, Nizhny Novgorod, Russia
E-mail address: medvedev@uic.nnov.ru

Dept. of Appl. Math., Nizhny Novgorod State Technical University, Nizhny Novgorod, Russia
E-mail address: zhuzhoma@mail.ru
Current e-mail address: zhuzhoma@maths.univ-rennes1.fr